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ON BROWNIAN EXCURSIONS IN LIPSCHITZ DOMAINS. PART I. LOCAL PATH PROPERTIES

KRZYSZTOF BURDZY AND RUTH J. WILLIAMS

ABSTRACT. A necessary and sufficient condition is given for a Brownian excursion law in a Lipschitz domain to share the local path properties with an excursion law in a halfspace. This condition is satisfied for all boundary points of every $C^{1,\alpha}$-domain, $\alpha > 0$. There exists a C^1-domain such that the condition is satisfied almost nowhere on the boundary. A probabilistic interpretation and applications to minimal thinness and boundary behavior of Green functions are given.

0. Introduction. We will show that the local path properties of Brownian excursions in $C^{1,\alpha}$-domains, $\alpha > 0$, are the same as the local path properties of Brownian excursions in a half-space (Theorem 3.1). This need not be the case if the domain is of class C^1 (Proposition 3.1). The method of proof is based on the exit system theory of Maisonneuve [**12**]. We will study Brownian excursion laws, i.e., one of the ingredients of an exit system. We will give a necessary and sufficient condition for an excursion law in a Lipschitz domain to have the same local path properties as an excursion law in a half-space (Theorems 2.1 and 2.2). This last result can be applied to obtain new criteria for minimal thinness (Theorem 4.1) and existence of a nondegenerate normal derivative for the Green function (Theorem 4.2).

Local path properties of 1-dimensional Brownian excursions have been known for some time. They are the same as the local path properties of the 3-dimensional Bessel process (see §2.10 of Itô and McKean [**11**] or §II.67 of Williams [**21**]). Dvoretsky and Erdös [**8**] and Shiga and Watanabe [**18**] have given Kolmogorov-type tests for Bessel processes. Local path properties of excursions of multidimensional Brownian motion have been studied by Shimura [**19**] (excursions of 2-dimensional Brownian motion from a line) and Burdzy [**1**] (n-dimensional case, $n \geq 2$).

Our theorem about minimal thinness generalizes results of Essen and Jackson [**9**] and Burdzy [**3, 4**]. A new criterion for the existence of the nondegenerate normal derivative for the Green function extends some results of Widman [**20**], Rippon [**17**] and Burdzy [**4**].

The present article mainly uses ideas from excursion theory (see Maisonneuve [**12**], Williams [**21**] and Burdzy [**1**]) and potential theory (see Doob [**7**]).

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1. Notation, definitions and review of relevant results. We will work with the n-dimensional Euclidean space $\mathbf{R}^n$, $n \geq 2$ and will use different Cartesian coordinate systems CS_1, CS_2, etc. The following notation will be used: $x = (x_1, \ldots, x_n)$, $\mathbf{R}_k^{n-1} = \{x \in \mathbf{R}^n : x_1 = 0 \text{ in } \mathrm{CS}_k\}$, $D_*^k = \{x \in \mathbf{R}^n : x_1 > 0 \text{ in } \mathrm{CS}_k\}$. If the coordinate system is fixed and no confusion may arise we will use symbols $\mathbf{R}^{n-1}$ and D_*. We will let $\tilde{x} = \mathrm{proj}_{\mathbf{R}_k^{n-1}} x$. We will make sure that it is clear which coordinate system is meant when the notation $(x_1, \ldots, x_n)$ or $\tilde{x}$ is used. For a set $A \subset \mathbf{R}^n$, we will use A^c to denote $\mathbf{R}^n \backslash A$.

We will say that a function $h: \mathbf{R}^{n-1} \to \mathbf{R}$ represents a domain D near $x \in \partial D$ in CS_k if there exists a neighborhood U of x such that

$$\{x \in \mathbf{R}^n : x_1 > h(\tilde{x}) \text{ in } \mathrm{CS}_k\} \cap U = D \cap U.$$

A domain will be called Lipschitz if it is represented by a Lipschitz function near every boundary point x (the function may depend on x). Similarly a domain will be called C^1 ($C^{1,\alpha}$) if it is represented near every boundary point by a function which has continuous (α-Hölder continuous) first order partial derivatives. Dahlberg [**6**] proved that the surface area measure and harmonic measure are mutually absolutely continuous in Lipschitz domains. The expression "almost all" will refer to either of these measures. The surface area measure on hyperplanes will be denoted dx. We will use the abbreviations $a^+ = \max(0, a)$ and $a^- = -\min(0, a)$.

We will use the canonical probability space $(\Omega, \mathbf{F}, \{\mathbf{F}_t\}, X)$. Here Ω is the set of all paths (functions) $\omega: [0, \infty) \to \mathbf{R}^n \cup \{\delta\}$ which are $\mathbf{R}^n$-valued and continuous on $[0, R)$, and satisfy $\omega(t) = \delta$ for all $t \geq R$. The lifetime $R = R(\omega)$ of a path is the time of the jump to the isolated trap δ in $\mathbf{R}^n \cup \{\delta\}$ (possibly $R = \infty$). The process X is defined by $X_t(\omega) = \omega(t)$ and associated raw σ-fields are defined by

$$\mathbf{F}_t^0 = \bigcap_{u > t} \sigma\{X(s), 0 \leq s \leq u\},$$
$$\mathbf{F}^0 = \sigma\{X(s), 0 \leq s < \infty\}.$$

$\mathbf{F}$ is the universal completion of $\mathbf{F}^0$ and $\mathbf{F}_t$ is the universal completion of $\mathbf{F}_t^0$ in $\mathbf{F}$. θ will denote the usual shift operator.

A random variable T is called a stopping time if $\{T \leq t\} \in \mathbf{F}_t$ for all $t \geq 0$. The pre-T σ-field $\mathbf{F}_T$ is defined by

$$\mathbf{F}_T = \{A \in \Omega : A \cap \{T \leq t\} \in \mathbf{F}_t \text{ for all } t \geq 0\}.$$

The hitting time T_A of a set A will be defined as $\inf\{t > 0 : X_t \in A\}$ where $\inf \phi = \infty$.

The distribution of standard Brownian motion starting from x will be denoted by P^x. The distribution of Brownian motion starting from x and killed on the boundary of a domain D will be denoted by P_D^x.

DEFINITION 1.1. A σ-finite measure H^x, $x \in \partial D$ on $(\Omega, \mathbf{F})$ is called an excursion law in D if $H^x(X(0) \neq x) = 0$ and H^x is strong Markov for the transition probabilities of Brownian motion in D in the following sense. For all H^x-a.s. strictly positive stopping times T, $\mathbf{F}_T$-measurable nonnegative a, and $\mathbf{F}$-measurable nonnegative b, we have:

$$H^x(a \cdot b(\theta_T)) = H^x(a \cdot P_D^{X_T}(b)).$$

There is no natural normalization of infinite measures so "unique" excursion law will mean "unique up to a multiplicative constant" excursion law. The expression $H_1^x = H_2^x$ will be understood in the same spirit.

A set $A \subset \mathbf{R}^n$ will be called regular for an excursion law H^x if $H^x(T_A > 0) = 0$. We will use the probabilistic expression H^x-a.s. instead of H^x-a.e.

DEFINITION 1.2. An excursion law H^x in a domain D will be called standard if $0 < H^x(T_B < \infty) < \infty$ for every nonempty open ball $B \subset \overline{B} \subset D$.

THEOREM 1.1. *Suppose that D is Lipschitz and $x \in \partial D$. Then there exists a unique standard excursion law H^x in D.*

PROOF. See Theorem 4.1 and Corollary 4.1 of Burdzy [**1**]. □

The standard excursion law in D_*^k starting from $x \in \mathbf{R}_k^{n-1}$ will be usually denoted by H_*^x.

Consider two domains D_1 and D_2 such that $x \in \partial D_1 \cap \partial D_2$. Suppose H^x is an excursion law in D_1. Kill H^x-excursions at $T_{D_2^c}$ on the set where $R \geq T_{D_2^c}$. On the set where $R < T_{D_2^c}$ continue the H^x-excursions as an independent Brownian motion in D_2 starting from $X(R-)$.

DEFINITION 1.3. The distribution of paths obtained by killing and continuing as described above will be called D_1-D_2 KAC-H^x.

By the 0-0 law for standard excursion laws (Lemma 4.1 of Burdzy [**1**]) we have either $T_{D_2^c} = 0$ H^x-a.s. or $T_{D_2^c} > 0$ H^x-a.s. provided H^x is standard. Thus if H^x is standard then either D_1-D_2KAC-H^x is concentrated on $\{\delta\}$ or it is an excursion law in D_2. The excursion law D_1-D_2 KAC-H^x need not be standard even if H^x is standard (see Remark 5.1(ii) of Burdzy [**2**]).

DEFINITION 1.4. Excursion laws H_1^x and H_2^x (not necessarily in the same domain) are said to share the local properties if they are mutually absolutely continuous on $[0,T)$ for some stopping time T which is strictly positive H_1^x-a.s. and H_2^x-a.s.

If excursion laws H_1^x and H_2^x are such that $H_2^x = D_1$-D_2 KAC-H_1^x and H_1^x is standard then they share the local properties (use $T = \min(R, T_{D_1^c}, T_{D_2^c})$). On the other hand, suppose H_1^x and H_2^x are standard excursion laws in domains D_1 and D_2, respectively, and they share the local properties. Then for $D_3 \equiv D_1 \cap D_2 \neq \emptyset$, $H_3^x \equiv D_1$-D_3 KAC -H_1^x is a standard excursion law in D_3, and D_3-D_2 KAC-$H_3^x = H_2^x$. It follows from this that D_1-D_2 KAC-$H_1^x = H_2^x$.

DEFINITION 1.5. An excursion law H^x is called locally flat if $H^x = D_*$-D KAC-H_*^x, where D is a domain and H_*^x is the standard excursion law in a half-space D_*.

We do not require in the above definition that H^x is standard.

Fix a Lipschitz region D and define excursions $\{e_t(s), s \geq 0\}$ of Brownian motion in D as follows. For each $t \geq 0$ such that $X(t) \in D^c$, let

$$e_t(s) = \begin{cases} X(t+s) & \text{if } T_{D^c} \circ \theta_t > s, \\ \delta & \text{otherwise.} \end{cases}$$

The above definition allows for constant excursions $e_t(s) \equiv \delta$. The following "exit system" formula is a special case of results of Maisonneuve [**12**]. Here L_t is a local time on ∂D of Brownian motion, i.e., a continuous additive functional with 1-potential $E^{\cdot}(e^{-T_{\partial D}})$, and $\tau_t = \inf\{s \geq 0 \colon L_s > t\}$.

THEOREM 1.2. *There exists a universally measurable family* $\{H^x\}, x \in \mathbf{R}^n$ *of* σ*-finite measures such that*

$$
\begin{aligned}
E^{\cdot}\left(\sum_{\substack{0<u<\infty \\ X_u \in \partial D}} Z^{\cdot}_u (f \circ e_u)\right) &= E^{\cdot}\left(\int_0^\infty Z_s H^{X_s}(f)\, dL_s\right) \\
&= E^{\cdot}\left(\int_0^{L_\infty} Z_{\tau(u)} H^{X_{\tau(u)}}(f)\, du\right)
\end{aligned}
\tag{1.1}
$$

for all $\{\mathbf{F}_t\}$*-predictable processes* Z *and universally measurable functions* f *on* Ω, *such that* f *is zero on constant excursions equal to* δ. *For each* $z \in \partial D$, *either* H^z *is a standard excursion law in* D *or* $H^z \equiv 0$.

Every pair (dL, H) satisfying the above theorem will be called an exit system in D.

REMARK 1.1. Suppose that some property holds H^x-a.s. for μ-almost all x, where μ is the measure associated with L (see Revuz [**15**, **16**]). Then the exit system formula shows that for each $z \in \mathbf{R}^n$, P^z-a.s. the excursions of Brownian motion in D have the same property.

2. Criteria for local flatness of excursion laws. Suppose that a Lipschitz function h represents the boundary of a domain D near the point $0 \in \partial D$. Let $D_* = \{x \in \mathbf{R}^n : x_1 > 0\}$ and H^0_* be the standard excursion law in D_*.

THEOREM 2.1. *The standard excursion law* H^0 *in* D *shares the local properties with* H^0_* *if and only if*

$$
\int_{\mathbf{R}^{n-1}\cap\{|x|\le 1\}} \frac{|h(x)|}{|x|^n}\, dx < \infty. \tag{2.1}
$$

PROOF. (i) If

$$
\int_{\mathbf{R}^{n-1}\cap\{|x|\le 1\}} \frac{h^+(x)}{|x|^n}\, dx = \infty \tag{2.2}
$$

then H^0_*-excursions hit ∂D immediately. This follows from Theorem 3.2 and Remark 3.2 of Burdzy [**3**]. Therefore H^0 and H^0_* do not share the local properties in this case.

(ii) Assume now that (2.1) holds. It follows that

$$
\int_{\mathbf{R}^{n-1}\cap\{|x|\le 1\}} \frac{h^-(x)}{|x|^n}\, dx < \infty. \tag{2.3}
$$

Define $D_1 = D_* \cup D$. Let H^0_1 be the standard excursion law in D_1. By Theorem 4.5 of Burdzy [**1**] and Corollary 4.2 of Burdzy [**4**], (2.3) implies that the excursion laws H^0_1 and H^0_* share the local properties. Let $K = D_* \backslash D = D_1 \backslash D$. In the present case (2.2) is not satisfied, so Theorem 3.2 and Remark 3.2 of Burdzy [**3**] imply that $H^0_*(T_K = 0) = 0$ and therefore $H^0_1(T_K = 0) = 0$. It follows that D_1-$D\,$KAC-H^0_1 is a standard excursion law in D which shares the local properties with H^0_1 and therefore with H^0_*. By the uniqueness of the standard excursion law we have $H^0 = D_1$-$D\,$KAC-H^0_1 and hence the "if" part of the theorem holds.

(iii) Suppose that (2.1) does not hold. Then either (2.2) holds and by (i), H^0 and H^0_* do not share the local properties, or both (2.2) and (2.3) fail to hold. Thus, to complete the proof of the theorem, we assume (2.2) and (2.3) do not hold. Let $D_2 = D \cap D_*$ and H^0_2 be the standard excursion law in D_2. We will prove that $H^0_3 \stackrel{\text{df}}{=} D_2\text{-}D\,\text{KAC-}H^0_2$ is not standard and show that this implies that H^0 and H^0_* do not share the local properties. This part of the proof is the longest and most complicated.

Suppose that h is Lipschitz with constant K, i.e., $|h(z) - h(y)| \le K|z-y|$. For $x \in D_*$, let $\hat{x} = (-x_1/4, x_2, \ldots, x_n)$, $r_1 = \min(x_1/16, x_1/8K)$ and $r_2 = 2r_1$. Let

$$B(y,r) = \{z \in \mathbf{R}^n : |z-y| = r\}.$$

Define $C_1(x) = B(x, r_1)$, $C_2(x) = B(\hat{x}, r_1)$, $C_3(x) = B(x, r_2)$ and $C_4(x) = B(\hat{x}, r_2)$. Let $M(x)$ be the interior of the smallest rectangle with sides parallel to the axes which contains $C_3(x)$ and $C_4(x)$. Since h represents ∂D near 0 it is easy to see that we may choose $\varepsilon > 0$ so small that $M(x) \subset D$ for all $x \in Q$, where $Q = \{x \in D : x_1 = -h(\tilde{x}) > 0 \text{ and } |x| \le \varepsilon\}$. For a fixed $t > 0$ define $T_1(t) = T_1 = t + T_Q \circ \theta_t$, and for $k \ge 1$, define Q_k and T_{k+1} inductively by

$$Q_k(t) = Q_k = Q \Big\backslash \bigcup_{i \le k} M(X(T_i)) \quad \text{if } T_k < \infty,$$

$$T_{k+1}(t) = T_{k+1} = \begin{cases} T_k + T_{Q_k} \circ \theta_{T_k} & \text{if } T_k < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

Let $B \subset D_2$ be a nonempty open ball. Then $H^0_2(T_B < \infty) > 0$. Define events

$$A_k(t) = A_k = \{T_k < \infty, T_B \circ \theta_{T_k} < \infty\}.$$

Fix an arbitrary integer $m > 0$ and an arbitrary real number $\varepsilon_0 > 0$. The excursion law H^0_2 shares the local properties with H^0_*. Since (2.3) holds with h^+ in place of h^-, this follows from Theorem 4.5 of Burdzy [**1**] and Corollary 4.2 of Burdzy [**4**]. We have assumed that

$$\int_{\mathbf{R}^{n-1} \cap \{|x| \le 1\}} \frac{h^-(x)}{|x|^n} dx = \infty.$$

By Lemma 3.1 of Burdzy [**3**] and Example 1.XII.12(b) of Doob [**7**] there is a truncated cone

$$V = \{x \in \mathbf{R}^n : |x| < \varepsilon,\ x_1 > a|\tilde{x}|\} \subset D_* \backslash Q$$

for some $a > 0$, $\varepsilon > 0$, such that $H^0_*(T_V \ne 0) = 0$. Thus, H^0_*-a.s., excursions immediately hit the set V above Q and by Theorem 3.2 and Remark 3.2 of Burdzy [**3**] they immediately hit a set below Q, so it follows that $H^0_*(T_Q \ne 0) = 0$ and therefore $H^0_2(T_Q \ne 0) = 0$. Hence, for each fixed k, $1_{A_k(t)} \to 1_{\{T_B < \infty\}}$ H^0_2-a.s. as $t \to 0$. Combining this with $1_{A_k(t)} \le 1_{\{T_B < \infty\}}$, it follows that $H^0_2(A_k(t)) \to H^0_2(T_B < \infty)$ as $t \to 0$. Choose $t > 0$ so small that

$$H^0_2(A_k) \ge (1 - \varepsilon_0) H^0_2(T_B < \infty)$$

for all $k \le m$. Define events $\tilde{A}_k$ for $k \ge 1$ by

$$\begin{aligned} \tilde{A}_k = \{T_k < \infty \text{ and } S_k \stackrel{\text{df}}{=} T_k + T(C_2(X(T_k))) \circ \theta_{T_k} < T_k + T(\partial M(X(T_k))) \circ \theta_{T_k} \\ \text{and } T_B \circ \theta_{S_k} < T(\partial(M(X(T_k)) \cup D_2)) \circ \theta_{S_k}\}. \end{aligned}$$

For $z \in Q$, let

$$p = p(z) = P^z_{M(z)}(T_{C_2(z)} < \infty, T_{C_1(z)} \circ \theta(T_{C_2(z)}) < \infty).$$

Now $p > 0$ and by Brownian scaling, p does not depend on z. The Harnack inequality applied to the harmonic function $P^{\cdot}_{D_2}(T_B < \infty)$ in the interior of $C_3(x)$ shows that for all $y \in C_1(x)$

$$P^y_{D_2}(T_B < \infty) \geq c P^x_{D_2}(T_B < \infty)$$

where $c > 0$ does not depend on x or y. For $x \in Q$, define

$$\hat{T}(x) = \begin{cases} T_{C_2(x)} + T_{C_1(x)} \circ \theta(T_{C_2(x)}) & \text{if } T_{C_2(x)} < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

The strong Markov property applied at T_k and $T_k + \hat{T}(X(T_k)) \circ \theta_{T_k}$ implies that

$$\begin{aligned} H^0_3(\tilde{A}_k) &\geq \int_Q \int_{C_1(x)} P^y_{D_2}(T_B < \infty) P^x_{M(x)}(X(\hat{T}(x)) \in dy) H^0_3(X(T_k) \in dx) \\ &\geq \int_Q \int_{C_1(x)} P^y_{D_2}(T_B < \infty) P^x_{M(x)}(X(\hat{T}(x)) \in dy) H^0_2(X(T_k) \in dx) \\ &\geq \int_Q \int_{C_1(x)} c P^x_{D_2}(T_B < \infty) P^x_{M(x)}(X(\hat{T}(x)) \in dy) H^0_2(X(T_k) \in dx) \\ &= \int_Q c P^x_{D_2}(T_B < \infty) p H^0_2(X(T_k) \in dx) \\ &= cpH^0_2(A_k) \geq cp(1 - \varepsilon_0) H^0_2(T_B < \infty). \end{aligned}$$

The last inequality holds for a fixed $t = t(\varepsilon_0, m)$ and all $k \leq m$. For t fixed, the events $\tilde{A}_k$, $k = 1, \ldots, m$, are disjoint and therefore

$$H^0_3(T_B < \infty) \geq \sum_{k=1}^{m} H^0_3(\tilde{A}_k) \geq mcp(1 - \varepsilon_0) H^0_2(T_B < \infty).$$

The above formula holds for arbitrary m and ε_0 so $H^0_3(T_B < \infty) = \infty$, i.e., H^0_3 is not standard.

Suppose now that H^0_* and H^0 share the local properties. Then

$$H^0(T_{\mathbf{R}^{n-1}} = 0) = 0$$

and $H^0_4 \stackrel{\text{df}}{=} D\text{-}D_2\,\text{KAC-}H^0$ is a standard excursion law in D_2. One has therefore $H^0 = D_2\text{-}D\,\text{KAC-}H^0_4$. By the uniqueness of the standard excursion laws we have $H^0_4 = H^0_2$ and $H^0 = D_2\text{-}D\,\text{KAC-}H^0_2 = H^0_3$. It has been proved that H^0_3 is not standard which contradicts the assumption that H^0 is standard. This contradiction proves that H^0 and H^0_* do not share the local properties. □

REMARKS 2.1. (i) In part (iii) above it has been proved implicitly that $H^0(T_{\mathbf{R}^{n-1}} \neq 0) = 0$. This property is in fact equivalent to $R = L_{\mathbf{R}^{n-1}}$ for the Brownian motion in D starting at a point $x \in D$ and conditioned to converge to $0 \in \partial D$. Here $L_{\mathbf{R}^{n-1}}$ denotes the last exit time from $\mathbf{R}^{n-1}$. To prove that $R = L_{\mathbf{R}^{n-1}}$ for the conditioned Brownian motion one may use a Borel-Cantelli Lemma for slightly dependent events (see for example Proposition 3.1 of Port and Stone [**14**]). This requires finding a suitable sequence of events $\{B_k\}$ replacing $\{\tilde{A}_k\}$ in the above proof and finding probabilities of B_k's and their intersections.

This seems like a difficult task in view of the fact that transition probabilities and hitting probabilities are not given explictly for conditioned Brownian motion in an arbitrary domain. The approach taken in the above proof is easier because: (i) it requires a sequence of disjoint events $\{\tilde{A}_k\}$ (instead of "almost independent" events $\{B_k\}$), and (ii) transition and hitting probabilities for an excursion law correspond to Brownian motion (not conditioned Brownian motion). Thus it pays sometimes to use σ-finite measures instead of probability measures.

(ii) The method of proof applied in part (iii) above was used earlier in Theorem 7.1 of Burdzy [**5**] to obtain a result about angular derivatives.

Suppose that a Lipschitz function h represents D near a point $x_0 \in \partial D$ in a coordinate system CS_1. By first performing a translation of coordinates, we may and do assume that $x_0 = 0$ in CS_1. Let H^{x_0} be the standard excursion law from x_0 in D.

THEOREM 2.2. *The excursion law H^{x_0} is locally flat if and only if the function h has a total derivative at $0 \in \mathbf{R}_1^{n-1}$ and*

$$\int_{\mathbf{R}_1^{n-1}\cap\{|x|\le 1\}} \frac{|h(x) - \nabla h(0)\cdot x|}{|x|^n}\,dx < \infty. \tag{2.4}$$

Suppose that $x_0 = 0$ in a coordinate system CS_2 and $D_*^2 = \{x : x_1 > v\cdot\tilde{x} \text{ in } \mathrm{CS}_1\}$ for some vector $v \in \mathbf{R}_1^{n-1}$. If h is a C^1 function, then for $v = \nabla h(0)$, there is a C^1 function g representing D in CS_2 near x_0, and one can apply Theorem 2.1 to g to deduce Theorem 2.2. However, if h is merely Lipschitz (or differentiable but not continuously so), then D need not be representable in CS_2 near 0, by *any* function, Lipschitz or not. For this reason, we prove the following lemma before Theorem 2.2.

LEMMA 2.1. (i) *If*

$$\int_{\mathbf{R}_1^{n-1}\cap\{|x|\le 1\}} \frac{(h(x) - v\cdot x)^+}{|x|^n}\,dx < \infty \tag{2.5}$$

then there exists a nonnegative Lipschitz function g_1 which represents a domain D_1 near x_0 in CS_2 and such that for some neighborhood U of x_0 we have

$$D_1 \cap U \subset D \cap U \quad \textit{and} \quad \int_{\mathbf{R}_2^{n-1}\cap\{|x|\le 1\}} \frac{g_1(x)}{|x|^n}\,dx < \infty.$$

(ii) *If*

$$\int_{\mathbf{R}_1^{n-1}\cap\{|x|\le 1\}} \frac{(h(x) - v\cdot x)^+}{|x|^n}\,dx = \infty \tag{2.6}$$

then there exists a nonnegative Lipschitz function g_2 which represents a domain D_2 near x_0 in CS_2 and such that for some neighborhood U of x_0 we have

$$D \cap U \cap D_*^2 \subset D_2 \cap U \cap D_*^2$$

and

$$\int_{\mathbf{R}_2^{n-1}\cap\{|x|\le 1\}} \frac{g_2(x)}{|x|^n}\,dx = \infty.$$

(iii) *If*

$$\int_{\mathbf{R}_1^{n-1}\cap\{|x|\le 1\}} \frac{(h(x)-v\cdot x)^-}{|x|^n}\,dx<\infty \tag{2.7}$$

then there exists a nonpositive Lipschitz function g_3 which represents a domain D_3 near x_0 in CS_2 and such that for some neighborhood U of x_0 we have

$$D\cap U\subset D_3\cap U \quad \textit{and} \quad \int_{\mathbf{R}_2^{n-1}\cap\{|x|\le 1\}} \frac{|g_3(x)|}{|x|^n}\,dx<\infty.$$

PROOF. The proofs of the three assertions are totally elementary and very similar to one another so we will sketch only the proof of the existence of g_1.

Assume that (2.5) holds. This implies that

$$\lim_{|x|\to 0}(h(x)-v\cdot x)^+|x|^{-1}=0. \tag{2.8}$$

To see this suppose that

$$\limsup_{|x|\to 0}(h(x)-v\cdot x)^+|x|^{-1}=a>0.$$

This implies that there exists a sequence $x_m\in\mathbf{R}_1^{n-1}$ and $m\ge 1$ such that $(h(x_m)-v\cdot x_m)^+|x_m|^{-1}>a/2$ and $2^{-k_m-1}\le|x_m|\le 2^{-k_m}$, where $\{k_m\}$ is an increasing sequence of integers such that $k_{m+1}-k_m\ge 3$. The function $(h(x)-v\cdot x)^+$ is Lipschitz with a constant $K_1>1$ so $(h(x)-v\cdot x)^+|x|^{-1}>a/16$ for all x in the set $W_m=\{z\in\mathbf{R}_1^{n-1}\colon |z-x_m|<r\}$ where $r=2^{-k_m}\min(a/8K_1,1/4)$. The volume of W_m is equal to $c\cdot r^{n-1}$ so we have

$$\begin{aligned}\int_{W_m}\frac{(h(x)-v\cdot x)^+}{|x|^n}\,dx&\ge\frac{a}{16}\cdot(2^{-k_m+1})^{-n+1}\cdot cr^{n-1}\\&=\frac{a}{16}\cdot(2^{-k_m+1})^{-n+1}\cdot c\left(2^{-k_m}\min\left(\frac{a}{8K_1},\frac14\right)\right)^{n-1}=c_1>0\end{aligned}$$

where c_1 does not depend on m. The sets W_m are disjoint so

$$\begin{aligned}\int_{\mathbf{R}_1^{n-1}\cap\{|x|\le1\}}\frac{(h(x)-v\cdot x)^+}{|x|^n}\,dx&\ge\sum_{m=1}^{\infty}\int_{W_m}\frac{(h(x)-v\cdot x)^+}{|x|^n}\,dx\\&\ge\sum_{m=1}^{\infty}c_1=\infty.\end{aligned}$$

This contradicts (2.5) and we conclude that (2.8) holds.

Choose j so large that

$$(h(x)-v\cdot x)^+|x|^{-1}<2^{-3}(n-1)^{-1/2} \tag{2.9}$$

for all x in the set

$$Q=\{x\in\mathbf{R}_1^{n-1}\colon |x_k|\le 2^{-j},k=2,\dots,n\}.$$

For $i>j$ define inductively sequences of sets $Q_{m_i+1},\dots,Q_{m_{i+1}}$ where $m_{j+1}=0$. For each $i>j$ let $Q_{m_i+1},\dots,Q_{m_{i+1}}$ be the sequence of all sets which satisfy the following conditions. If $m_i + 1 \le m \le m_{i+1}$ then

(a) Q_m has the form $Q_m = \{x \in Q: j_k 2^{-i} < x_k < (j_k + 1)2^{-i}, k = 2, \dots, n\}$ in CS_1 for some integers $j_2, \dots, j_n$,

(b) $0 \notin \partial Q_m$,

(c) $\sup_{x \in Q_m} (h(x) - v \cdot x)^+ > 2^{-i-3}$,

(d) the set Q_m is disjoint from all sets $Q_l, l \le m_i$.

Since the function $(h(x) - v \cdot x)^+$ is Lipschitz with the constant $K_1 > 1$ we have $(h(x) - v \cdot x)^+ \ge 2^{-i-4}$ for all x in the set $V_m = \{z \in Q_m: |z - \hat{x}| < 2^{-i-4}/K_1\}$ where $\hat{x} = \hat{x}(m)$ is a point in the closure of Q_m where the supremum in (c) is attained.

Let $a_m = \inf_{x \in Q_m} |x|$ and $b_m = \sup_{x \in Q_m} |x|$ and note that $b_m \le a_m 2(n-1)^{1/2}$ by (b). The volume of V_m is at least $(2^{-i-5}/K_1)^{n-1}$ and therefore

$$\begin{aligned}(2.10)\quad \int_{Q_m} \frac{(h(x) - v \cdot x)^+}{|x|^n} dx &\ge \int_{V_m} \frac{(h(x) - v \cdot x)^+}{|x|^n} dx \\ &\ge 2^{-i-4} (2^{-i-5}/K_1)^{n-1} b_m^{-n} \\ &\ge 2^{-i-4} (2^{-i-5}/K_1)^{n-1} a_m^{-n} 2^{-n} (n-1)^{-n/2} \\ &= (2^{-6n+1} K_1^{-n+1} (n-1)^{-n/2})(2^{-in} a_m^{-n}).\end{aligned}$$

Let

$$Q_m^1 = \{x \in \mathbf{R}_2^{n-1}: \mathrm{dist}(\tilde{x}, Q_m) \le 2^{-i-1}\}$$

and

$$Q_m^2 = \{x \in \mathbf{R}_2^{n-1}: \mathrm{dist}(\tilde{x}, Q_m) \le 2^{-i-2}\}$$

where $\tilde{x}$ is the projection in CS_1. The function

$$f_m(x) = \left[2^{-i-2} - \mathrm{dist}(x, Q_m^2)\right]^+$$

defined for $x \in \mathbf{R}_2^{n-1}$ is Lipschitz with constant 1. It follows from (2.9) and (c) that $\sup_{x \in Q_m} (h(x) - v \cdot x)^+ < 2^{-i-2}$ so

$$\begin{aligned}(2.11)\quad &\{x \in \mathbf{R}^n: v \cdot \tilde{x} < x_1 < h(\tilde{x}) \text{ and } \tilde{x} \in Q_m \text{ in } \mathrm{CS}_1\} \\ &\qquad \subset \{x \in \mathbf{R}^n: 0 < x_1 < f_m(\tilde{x}) \text{ in } \mathrm{CS}_2\}.\end{aligned}$$

The support of f_m is contained in Q_m^1 which has a volume no greater than $(|v|^2 + 1)^{1/2} (2^{-i+1})^{n-1}$. Thus

$$\begin{aligned}\int_{\mathbf{R}_2^{n-1}} \frac{f_m(x)}{|x|^n} dx &\le 2^{-i-2} (|v|^2 + 1)^{1/2} (2^{-i+1})^{n-1} (a_m/2)^{-n} \\ &= 2^{2n-3} (|v|^2 + 1)^{1/2} (2^{-in} a_m^{-n}).\end{aligned}$$

This and (2.10) show that

$$\int_{\mathbf{R}_2^{n-1}} \frac{f_m(x)}{|x|^n} dx \le c \int_{Q_m} \frac{(h(x) - v \cdot x)^+}{|x|^n} dx$$

where c does not depend on m. Define $g_1 = \sup_k f_k$. Since

$$\{x \in Q: (h(x) - v \cdot x)^+ > 0\} \subset \bigcup_{k=1}^{\infty} \overline{Q_k}$$

it follows from (2.11) that $D_1 = \{x\colon x_1 > g_1(\tilde{x}) \text{ in } \mathrm{CS}_2\}$ satisfies the condition stated in the theorem. We have

$$\int_{\mathbf{R}_2^{n-1}} \frac{g_1(x)}{|x|^n}dx \le \sum_{k=1}^{\infty} \int_{\mathbf{R}_2^{n-1}} \frac{f_k(x)}{|x|^n}dx \le c\sum_{Q_k}\int_{Q_k} \frac{(h(x)-v\cdot x)^+}{|x|^n}dx$$
$$\le c\int_{\mathbf{R}_1^{n-1}\cap\{|x|\le 1\}} \frac{(h(x)-v\cdot x)^+}{|x|^n}dx < \infty. \quad \square$$

PROOF OF THEOREM 2.2. (i) Suppose that (2.4) holds. Let the coordinate system CS_2 in Lemma 2.1 be defined such that $v = \nabla h(0)$. It follows from (2.4) that (2.5) and (2.7) are satisfied. Define D_1 and D_3 as in (i) and (iii) of Lemma 2.1. Choose $\varepsilon > 0$ so small that $D_1 \cap U \subset D \cap U \subset D_3 \cap U$ where $U = \{z\colon |z - x_0| < \varepsilon\}$. Define $D_4 = D\backslash(\overline{U}\backslash D_1)$ and $D_5 = D \cup (U \cap D_3)$. If $H_*^{x_0}$ is a standard excursion law in D_*^2 then by Lemma 2.1 and Theorem 2.1

$$H_4^{x_0} = D_*^2\text{-}D_4\mathrm{KAC}\text{-}H_*^{x_0} \quad \text{and} \quad H_5^{x_0} = D_*^2\text{-}D_5\,\mathrm{KAC}\text{-}H_*^{x_0}$$

are standard excursion laws. Let $H_1^{x_0} = D_*^2\text{-}D\,\mathrm{KAC}\text{-}H_*^{x_0}$. By definition, the excursion law $H_1^{x_0}$ is locally flat.

Fix a nonempty open ball $B \subset D_4$. We have $D_4 \subset D \subset D_5$ and

$$0 < H_4^{x_0}(T_B < \infty) \le H_1^{x_0}(T_B < \infty) \le H_5^{x_0}(T_B < \infty) < \infty$$

so $H_1^{x_0}$ is a standard excursion law in D and therefore $H_1^{x_0} = H^{x_0}$. It follows that H^{x_0} is locally flat.

(ii) Assume that H^{x_0} is locally flat. Suppose that H^{x_0} shares the local properties with the standard excursion law $H_*^{x_0}$ in D_*^2, where D_*^2 is the upper half-space in some coordinate system CS_2. We can assume that $x_0 = 0$ in CS_2 and that D_*^2 has the form $\{x\colon x_1 > v\cdot\tilde{x} \text{ in } \mathrm{CS}_1\}$. We will prove that $\mathbf{R}_2^{n-1}$ is a tangent to ∂D at x_0, that $\nabla h(0)$ exists and equals v, and (2.4) holds.

Suppose that (2.6) is satisfied and let D_2 be defined as in part (ii) of Lemma 2.1. It follows from part (i) of the proof of Theorem 2.1 and Lemma 2.1(ii) that

$$H_*^{x_0}(T_{\partial D_2} \ne 0) = 0. \tag{2.12}$$

For some neighborhood U of x_0, $\partial D_2 \cap U \cap D \cap D_*^2 = \varnothing$ and H^{x_0}-excursions stay in $U \cap D \cap D_*^2$ for some interval of time $(0,T)$, $T > 0$ and therefore

$$H^{x_0}(T_{\partial D_2} = 0) = 0.$$

This contradicts (2.12) since the excursion laws H^{x_0} and $H_*^{x_0}$ share the local properties. Thus (2.6) cannot be satisfied and therefore (2.5) holds.

Suppose that (2.5) holds and the integral in (2.7) is infinite. Repeat part (iii) of the proof of Theorem 2.1 with the following changes:

(a) redefine D_2 as $D \cap D_*^2$,

(b) for $x \in D_*^2$ redefine $\hat{x}$, r_1, r_2 and Q as follows:

$$\hat{x} = \{v\cdot\tilde{x} - (x_1 - v\cdot\tilde{x})/4, x_2, \dots, x_n\} \text{ in } \mathrm{CS}_1,$$
$$r_1 = (x_1 - v\cdot\tilde{x})\cdot\min(1/16, 1/(8K)) \text{ in } \mathrm{CS}_1,$$
$$r_2 = 2r_1,$$
$$Q = \{x \in D\colon x_1 = 2v\cdot\tilde{x} - h(\tilde{x}) > v\cdot\tilde{x} \text{ and } |x| \le \varepsilon \text{ in } \mathrm{CS}_1\},$$

where $\varepsilon > 0$ is a suitable constant.

With these modifications part (iii) of the proof of Theorem 2.1 shows that H^{x_0} and $H_*^{x_0}$ do not share the local properties. This contradiction means that the negation of (2.7) cannot hold. We conclude that (2.5) and (2.7) are satisfied.

As in the proof of Lemma 2.1, it follows from (2.5) and (2.7) that

$$\lim_{|x|\to 0} |h(x) - v \cdot \tilde{x}| \cdot |x|^{-1} = 0.$$

Thus $\nabla h(0) = v$ and (2.5) and (2.7) imply (2.4). $\square$

REMARK 2.2. Theorems 2.1 and 2.2 generalize Theorem 4.3 of Burdzy [**1**].

3. Brownian excursions in smooth domains.

LEMMA 3.1. *If D is a $C^{1,\alpha}$-domain for some $\alpha > 0$ then the condition* (2.4) *is satisfied for every boundary point.*

PROOF. Let D be a $C^{1,\alpha}$-domain for some $\alpha > 0$ and $x \in \partial D$. Choose a coordinate system CS_1 such that $x = 0$ in CS_1, the boundary of D is represented by a function h near 0 and $\nabla h(0) = 0$. We have $|h(x)| \le c|x|^{1+\alpha}$ for some $c > 0$ and all x in $\mathbf{R}^{n-1} \cap \{|x| \le 1\}$, by the assumption that the domain is $C^{1,\alpha}$. Thus,

$$\int_{\mathbf{R}_1^{n-1}\cap\{|x|\le 1\}} \frac{|h(x)|}{|x|^n} dx \le \int_{\mathbf{R}_1^{n-1}\cap\{|x|\le 1\}} \frac{c|x|^{1+\alpha}}{|x|^n} dx < \infty$$

and (2.4) is satisfied. $\square$

Let $D \subset \mathbf{R}^n$, $n \ge 2$ be a $C^{1,\alpha}$-domain for some $\alpha > 0$ and let L be a continuous additive functional such that its associated measure is the surface area measure on ∂D (for the existence of such an L, see Revuz [**15**, **16**]).

For each point $x \in \partial D$ construct an excursion law H^x in D as follows. Choose a coordinate system $\mathrm{CS}_1 = \mathrm{CS}_1(x)$ as in the proof of Lemma 3.1. Let H_*^x be the standard excursion law in D_*^1 starting from x and normalized so that $H_*^x(T_B < \infty) = 1$, where $B = \{x : x_1 = 1 \text{ in } \mathrm{CS}_1\}$. (This is possible because $0 < H_*^x(T_B < \infty) < \infty$ by Theorem 3.3(iii) of Burdzy [**1**].) Define $H^x = D_*^1\text{-}D\,\mathrm{KAC}\text{-}H_*^x$.

THEOREM 3.1. *The pair (dL, H) is an exit system in D. All excursion laws H^x, $x \in \partial D$ are standard and locally flat.*

PROOF. All excursion laws H^x, $x \in \partial D$ are standard and locally flat by Lemma 3.1 and Theorem 2.2. The pair (dL, H) is an exit system in D by Theorems 4.4 and 5.1 of Burdzy [**1**]. Note that the crucial assumption (D) of Theorem 4.4 is satisfied due to Lemma 3.1 proved above. $\square$

REMARKS 3.1. (i) The above theorem generalizes Theorem 5.1 of Burdzy [**1**] where the same conclusion was reached under the assumption that D is a C^2-domain. The theorem cannot be further strengthened to C^1-domains as shown by the next proposition.

(ii) The above theorem holds under the assumption that D or $\mathbf{R}^n \backslash D$ is convex, with no additional assumptions about the regularity of ∂D. See Theorem 5.1 of Burdzy [**1**] for a proof of this statement. Alternatively, in this case one can show that (2.4) is satisfied almost everywhere on the boundary of D.

(iii) In view of Remark 1.1, we see that for each $z \in \mathbf{R}^n$, P^z-a.s. the excursions of Brownian motion in $C^{1,\alpha}$-domains share the local path properties with

Brownian excursions from hyperplanes. See Theorems 3.1, 3.2 and 3.3 of Burdzy [**1**] (especially Theorem 3.3(vii) and (viii)) for some path properties of excursions from a hyperplane.

PROPOSITION 3.1. *There exists a C^1-domain D in $\mathbf{R}^2$ such that* (2.4) *is satisfied for almost no $x \in \partial D$. There exists an exit system (dL, H) in D such that no excursion law H^x is locally flat.*

PROOF. We will prove first that for every integer $k > 0$ there exists a nondecreasing differentiable function $g_k\colon [0,1] \to \mathbf{R}$ such that $0 \le g_k'(x) \le 2^{-k}$ for all x and the following condition holds for x in a set of measure greater than $1 - 2^{-k}$:

$$\int_0^1 \frac{|g_k(x) + g_k'(x)(y-x) - g_k(y)|}{|y-x|^2} dy \ge k. \tag{3.1}$$

We will define the function g_k'. This and the condition $g(0) = 0$ will uniquely define the function g_k. Let $a = 2^{-2}\exp(-k2^{2k+4})$, $b = a \cdot 2^{-k}$, $g_k'(1) = 0$, and for each integer $m \ge 0$, let

$$\begin{aligned} g_k'(x) &= 0 \qquad \text{if } x = m(a+b) \text{ or } x = m(a+b) + a, \\ &= 2^{-k} \quad \text{if } x = m(a+b) + a + b/2. \end{aligned}$$

Extend g_k' to the whole interval $[0,1]$ in such a way that it is continuous and linear on every interval where it has not been defined above.

Consider an $x \in [0, \frac{1}{2}]$ such that $g_k'(x) = 0$. Then for $y \in [x, x + \frac{1}{2}]$ we have

$$|g_k(x) + g_k'(x)(y-x) - g_k(y)| = |g_k(x) - g_k(y)| = \int_x^y g_k'(z)dz.$$

It is elementary to check that the last integral is greater than $2^{-2k-3}(y-x)$ for all $y \ge x + 2(a+b)$. Therefore

$$\begin{aligned} \int_0^1 \frac{|g_k(x) + g_k'(x)(y-x) - g_k(y)|}{|y-x|^2} dy &\ge \int_{x+2(a+b)}^{x+1/2} \frac{2^{-2k-3}(y-x)}{|y-x|^2} dy \\ &= 2^{-2k-3}(\ln \tfrac{1}{2} - \ln 2(a+b)) \ge 2^{-2k-3}(\ln \tfrac{1}{2} - \ln 4a) \\ &= 2^{-2k-3}(\ln \tfrac{1}{2} + k2^{2k+4}) = 2^{-2k-3}\ln \tfrac{1}{2} + 2k \ge k. \end{aligned}$$

It can be proved analogously that (3.1) holds for all $x \in [\frac{1}{2}, 1]$ such that $g_k'(x) = 0$. It is now easy to see from the definition of g_k' that (3.1) is satisfied for x in a set of measure greater than $1 - 2^{-k}$. The proof of existence of the functions g_k with the desired properties is complete.

For $x \in [0,1]$, let $h(x) = \sum_{k=1}^\infty g_k(x)$ and $h'(x) = \sum_{k=1}^\infty g_k'(x)$. It is easy to see that the functions h and h' are well defined and h' is the derivative of h. The function h' is continuous since the series $\sum_k g_k'$ of continuous functions converges uniformly. Thus h is a C^1-function.

Let $h_m = \sum_{k=1}^m g_k$. The derivative of h_m is piecewise linear so for x outside a finite set we have

$$|h_m(x) + h_m'(x)(y-x) - h_m(y)| \le c_m(x) \cdot |y-x|^2 \quad \text{for all } y \in [0,1]. \tag{3.2}$$

If (3.2) holds then

$$\int_0^1 \frac{|h_m(x) + h_m'(x)(y-x) - h_m(y)|}{|y-x|^2} dy \le \int_0^1 \frac{c_m(x)|y-x|^2}{|y-x|^2} dy = c_m(x).$$

For every integer $j > 0$ we will find a set $A_j \subset [0,1]$ such that the measure of A_j is greater than $1 - 2^{-j}$ and for all $x \in A_j$ we have

$$\int_0^1 \frac{|h(x) + h'(x)(y-x) - h(y)|}{|y-x|^2} dy > j. \tag{3.3}$$

Let $f_m = h - h_m$. The function g'_k is zero on a set of measure greater than $1 - 2^{-k}$. Therefore the function $f'_m = \sum_{k=m+1}^{\infty} g'_k$ is equal to zero on a set of measure greater than $1 - \sum_{k=m+1}^{\infty} 2^{-k} = 1 - 2^{-m}$. Let A_j be the set of all x's such that $f'_j(x) = 0$ and (3.2) holds with $m = j$. The measure of A_j is greater than $1 - 2^{-j}$ and we will now show that (3.3) holds for all $x \in A_j$.

Fix a point $x \in A_j$. Choose k so large that $k - c_j(x) > j$. Note that $g'_k(x) = 0$ and (3.1) holds for this choice of x and k. Since all functions g_m are nondecreasing and $k > j$ we have

$$|f_j(x) - f_j(y)| \geq |g_k(x) - g_k(y)|.$$

Thus

$$\begin{aligned}
&\int_0^1 \frac{|h(x) + h'(x)(y-x) - h(y)|}{|y-x|^2} dy \\
&= \int_0^1 \frac{|h_j(x) + f_j(x) + (h'_j(x) + f'_j(x))(y-x) - h_j(y) - f_j(y)|}{|y-x|^2} dy \\
&= \int_0^1 \frac{|h_j(x) + f_j(x) + h'_j(x)(y-x) - h_j(y) - f_j(y)|}{|y-x|^2} dy \\
&\geq \int_0^1 \frac{|f_j(x) - f_j(y)| - |h_j(x) + h'_j(x)(y-x) - h_j(y)|}{|y-x|^2} dy \\
&= \int_0^1 \frac{|f_j(x) - f_j(y)|}{|y-x|^2} dy - \int_0^1 \frac{|h_j(x) + h'_j(x)(y-x) - h_j(y)|}{|y-x|^2} dy \\
&\geq \int_0^1 \frac{|g_k(x) - g_k(y)|}{|y-x|^2} dy - c_j(x) \\
&= \int_0^1 \frac{|g_k(x) + g'_k(x)(y-x) - g_k(y)|}{|y-x|^2} dy - c_j(x) \\
&\geq k - c_j(x) > j.
\end{aligned}$$

Since j is arbitrary it follows that the set of x's for which the integral in (3.3) is finite must have measure zero.

Extend the function h to the whole real line by

$$h(x) = h(x - [x]) + [x] \cdot h(1)$$

where $[x]$ denotes the integer part of x. Let $D = \{x \in \mathbf{R}^2 : x_2 > h(x_1)\}$. It is easy to see that D is a C^1-domain. Since the integral in (3.3) is infinite for almost all $x \in [0,1]$ it follows that (2.4) is satisfied almost nowhere on the boundary of D.

Let (dL, H) be an exit system in D and let A be the set of all points $x \in \partial D$ such that H^x is locally flat. It follows from the first part of the proof that the set A has null harmonic measure so for each $z \in \mathbf{R}^n$, P^z-a.s. no excursion of Brownian motion has its endpoint in A. By time reversal, P^z-a.s. no excursion has its starting point in A. Define $H_1^x \equiv 0$ for $x \in A$ and $H_1^x = H^x$ otherwise. It is easy to see that (dL, H_1) is an exit system in D and no excursion law H_1^x is locally flat. □

REMARKS 3.2. (i) The above proposition implies by the exit system formula (see Remark 1.1) that for each $z \in \mathbf{R}^2$, P^z-a.s. no excursion of Brownian motion in the region constructed in the above proof shares all the local path properties with excursions from a line. The following example should clarify this remark.

If $e_t(\cdot)$ is a nonconstant excursion in the halfplane $D_* \subset \mathbf{R}^2$ (see §1 for the definitions) then $e_t(0) \in \mathbf{R}^1$ and the straight line $\mathbf{R}^1$ is not hit immediately by $\{e_t(s), s > 0\}$.

If $e_t(\cdot)$ is a nonconstant excursion in the region defined in the last proof then $\{e_t(s), s > 0\}$ immediately hits every straight line passing through $e_t(0)$. This holds, of course, for all excursions on P^z-almost all paths for all $z \in \mathbf{R}^2$. We omit the easy proof of this fact (see also Example 6.1 of Burdzy [**2**]).

(ii) Proposition 3.1 shows that quasi-locally flat excursion laws (terminology of Burdzy [**2**]) cannot be replaced by locally flat excursion laws in an exit system even if the region is quite smooth.

4. Applications to minimal thinness and boundary behavior of Green functions. The reader is referred to Doob [**7**] for definitions of Green function, Martin topology, minimal Martin boundary, minimal thinness, minimal fine topology and related concepts. We recall that the minimal Martin boundary and topology may be identified with the Euclidean boundary and topology in bounded Lipschitz domains (see Hunt and Wheeden [**10**]).

Suppose that a Lipschitz function h represents a region D near $0 \in \partial D$ in a coordinate system CS_1. Suppose that a hyperplane $\{x \in \mathbf{R}^n : x_1 = v \cdot \tilde{x} \text{ in } \mathrm{CS}_1\}$ is equal to $\mathbf{R}_2^{n-1}$ in a coordinate system CS_2.

THEOREM 4.1. (i) *Suppose that*

$$\int_{\mathbf{R}_1^{n-1} \cap \{|x| \le 1\}} \frac{(h(x) - v \cdot x)^-}{|x|^n} dx < \infty. \tag{4.1}$$

Then $D_*^2 \backslash D$ is minimal thin in $D \cup D_*^2$ at 0 if and only if

$$\int_{\mathbf{R}_1^{n-1} \cap \{|x| \le 1\}} \frac{(h(x) - v \cdot x)^+}{|x|^n} dx < \infty. \tag{4.2}$$

(ii) *Suppose that* (4.2) *holds. Then $D \backslash D_*^2$ is minimal thin in D at 0 if and only if* (4.1) *is satisfied.*

PROOF. (i) Suppose (4.1) holds. Let H_*^0 be the standard excursion law in D_*^2 and let $D_1 = D \cup D_*^2$. Define $H_1^0 = D_*^2\text{-}D_1\,\text{KAC-}H_*^0$. The excursion law H_1^0 is standard by Theorem 2.2. Let $D_2 = D \cap D_*^2$ and define $H_2^0 = D_*^2\text{-}D_2\,\text{KAC-}H_*^0$. Theorem 2.2 implies that H_2^0 is standard if and only if (4.2) is satisfied. In other words (4.2) holds if and only if $D_*^2 \backslash D_2$ is not regular for H_*^0. The excursion laws H_*^0 and H_1^0 share the local properties so (4.2) holds if and only if $D_*^2 \backslash D$ is not regular for H_1^0. This is equivalent to $D_*^2 \backslash D$ being minimal thin at 0 in D_1 by Lemma 3.1 of Burdzy [**3**].

(ii) Suppose that (4.1) and (4.2) hold. Then the standard excursion law H^0 in D is locally flat by Theorem 2.2. This excursion law shares the local properties with H^0_* and in particular $D\backslash D^2_*$ is not regular for H^0. Lemma 3.1 of Burdzy [**3**] implies that $D\backslash D^2_*$ is minimal thin in D at 0.

(iii) Suppose now that (4.2) holds but (4.1) is not satisfied. We will assume that $D\backslash D^2_*$ is not regular for H^0 and will show that this assumption leads to a contradiction. Indeed, under this assumption $H^0_3 \stackrel{\text{df}}{=} D\text{-}D_2\,\text{KAC-}H^0$ is a standard excursion law in D_2. Therefore $H^0_3 = H^0_2$ and so H^0_3 is locally flat. We have $H^0 = D_2\text{-}D\,\text{KAC-}H^0_3$ and it follows that H^0 is locally flat. This contradicts Theorem 2.2. We have shown that $D\backslash D^2_*$ is regular for H^0 and so by Lemma 3.1 of Burdzy [**3**], $D\backslash D^2_*$ is not minimal thin in D at 0. □

REMARK 4.1. The above theorem generalizes Theorem 3.2 of Burdzy [**3**] and Corollaries 4.2 and 4.3 of Burdzy [**4**] (see also Remark 4.2(ii) of the last paper). Alternative criteria for minimal thinness in a half-space were given earlier by Essen and Jackson [**9**].

For every domain $\tilde{D}$, extend the Green function $G_{\tilde{D}}(\cdot,\cdot)$ to $\mathbf{R}^n \times \mathbf{R}^n$ by declaring that $G_{\tilde{D}}(x,y) = 0$ for $(x,y) \notin \tilde{D} \times \tilde{D}$.

LEMMA 4.1. *Suppose that D_1 and D_2 are Lipschitz domains, $y \in D_1 \subset D_2$ and $x \in \partial D_1 \cap \partial D_2$. Then $D_2\backslash D_1$ is minimal thin in D_2 at x if and only if*

$$\operatorname*{mf\text{-}lim}_{\substack{z\to x\\ z\in D_1}} \frac{G_{D_1}(z,y)}{G_{D_2}(z,y)} > 0.$$

PROOF. The limits

$$\operatorname*{mf\text{-}lim}_{\substack{z\to x\\ z\in D_1}} \frac{G_{D_1}(z,y)}{G_{D_2}(z,y)} \quad \text{and} \quad \limsup_{\substack{z\to x\\ z\in D_1}} \frac{G_{D_1}(z,y)}{G_{D_2}(z,y)}$$

exist and are equal by Theorem 1.XII.14 of Doob [**7**]. It follows from Theorem 11 of Naim [**13**] that the limits are greater than 0 if and only if $D_2\backslash D_1$ is minimal thin at x in D_2. □

LEMMA 4.2. *Assume the hypotheses of Lemma* 4.1 *hold and let f be a function on D_2. If $D_2\backslash D_1$ is minimal thin in D_2 at x then*

$$\operatorname*{mf\text{-}lim}_{\substack{z\to x\\ z\in D_1}} f(z) = \operatorname*{mf\text{-}lim}_{\substack{z\to x\\ z\in D_2}} f(z)$$

if at least one side is well defined.

PROOF. Let P^y_x denote the distribution of Brownian motion in D_2 starting from $y \in D_1$ and conditioned to converge to x at the lifetime. By Theorem 3.III.3 of Doob [**7**], the last exit time from ∂D_1 is strictly less than the lifetime of the process P^y_x-a.s. and therefore $P^y_x(T_{\partial D_1} = \infty) > 0$. Let $\tilde{P}^y_x$ be the distribution P^y_x conditioned by $\{T_{\partial D_1} = \infty\}$. Then $\tilde{P}^y_x$ is the distribution of Brownian motion in D_1 conditioned to converge to x. It follows that

$$P^y_x\left(\lim_{t\to R} f(X_t) > a\right) = \tilde{P}^y_x\left(\lim_{t\to R} f(X_t) > a\right)$$

since the above probabilities may take values 0 or 1 only (see 2.X.11(c) of Doob [**7**]). The last formula implies the lemma by Theorem 3.III.3 of Doob [**7**]. □

Recall the notation of Theorem 4.1. Let N be the unit normal vector to $\mathbf{R}_2^{n-1}$ that points into D_*^2. Fix a point $x \in D \cap D_*^2$. Let

$$\frac{\partial}{\partial N} G_D \equiv \frac{\partial}{\partial N} G_D(x,0) = \lim_{\varepsilon \to 0} \frac{G_D(x, 0+\varepsilon N)}{\varepsilon}$$

if the limit exists.

THEOREM 4.2. *Assume the following interior cone condition:*

$$\{x \in \mathbf{R}^n : x \cdot N > a|\tilde{x}| \text{ in } \mathrm{CS}_2\} \cap U \subset D \tag{4.3}$$

for some $a > 0$ and neighborhood U of 0 in $\mathbf{R}^n$.

(i) *Suppose that* (4.1) *holds. Then $\partial G_D/\partial N$ exists and $\partial G_D/\partial N < \infty$ and* (4.2) *is a necessary and sufficient condition for $\partial G_D/\partial N > 0$.*

(ii) *Suppose that* (4.2) *holds. Then $\partial G_D/\partial N$ exists and $\partial G_D/\partial N > 0$ and* (4.1) *is a necessary and sufficient condition for $\partial G_D/\partial N < \infty$.*

PROOF. We will use the notation from the proof of Theorem 4.1.

(i) Assume that (4.1) and (4.2) hold. Let $D_1 = D \cup D_*^2$. The set $D_1 \backslash D_*^2$ is minimal thin in D_1 at 0 by Theorem 4.1(ii) (replace D by D_1 there). It follows that

$$0 < \operatorname*{mf\text{-}lim}_{\substack{y \to 0 \\ y \in D_1}} \frac{G_{D_*^2}(x,y)}{G_{D_1}(x,y)} \leq 1 \tag{4.4}$$

by Lemmas 4.1 and 4.2 and the fact that $G_{D_*^2}(x,y) \leq G_{D_1}(x,y)$. The set $D_*^2 \backslash D$ is minimal thin in D_1 at 0 by Theorem 4.1(i). Therefore Lemmas 4.1 and 4.2 imply that

$$\operatorname*{mf\text{-}lim}_{\substack{y \to 0 \\ y \in D_1}} \frac{G_D(x,y)}{G_{D_1}(x,y)} > 0. \tag{4.5}$$

This combined with (4.4) shows that

$$\operatorname*{mf\text{-}lim}_{\substack{y \to 0 \\ y \in D_1}} \frac{G_D(x,y)}{G_{D_*^2}(x,y)} > 0 \tag{4.6}$$

and by Lemma 4.2

$$\operatorname*{mf\text{-}lim}_{\substack{y \to 0 \\ y \in D_*^2}} \frac{G_D(x,y)}{G_{D_*^2}(x,y)} > 0. \tag{4.7}$$

This implies that

$$\lim_{\varepsilon \to 0} \frac{G_D(x, 0+\varepsilon N)}{G_{D_*^2}(x, 0+\varepsilon N)} > 0. \tag{4.8}$$

To see this, follow the proof of Theorem 1.XII.21 of Doob [**7**] and use assumption (4.3). It is elementary to check that

$$\lim_{\varepsilon \to 0} \frac{G_{D_*^2}(x, 0+\varepsilon N)}{\varepsilon} > 0 \tag{4.9}$$

and this combined with (4.8) shows that $\partial G_D/\partial N > 0$.

When (4.1) holds but (4.2) is not satisfied then the limits in (4.5), (4.6), (4.7) and (4.8) are equal to zero and $\partial G_D/\partial N = 0$.

(ii) Assume that (4.2) holds but (4.1) is not satisfied. Let $D_2 = D \cap D_*^2$. The set $D_*^2 \backslash D_2$ is minimal thin in D_*^2 at 0 by Theorem 4.1(i). Lemma 4.1 implies that

$$\text{mf-}\lim_{\substack{y\to 0\\ y\in D_2}} \frac{G_{D_2}(x,y)}{G_{D_*^2}(x,y)} > 0. \tag{4.10}$$

The set $D\backslash D_*^2$ and hence $D\backslash D_2$ is not minimal thin in D at 0, by Theorem 4.1(ii). Thus Lemma 4.1 implies that

$$\text{mf-}\lim_{\substack{y\to 0\\ y\in D_2}} \frac{G_{D_2}(x,y)}{G_D(x,y)} = 0$$

and this combined with (4.10) yields

$$\text{mf-}\lim_{\substack{y\to 0\\ y\in D_2}} \frac{G_{D_*^2}(x,y)}{G_D(x,y)} = 0.$$

Since $D_*^2\backslash D_2$ is minimal thin in D_*^2 at 0, it then follows from Lemma 4.2 that

$$\text{mf-}\lim_{\substack{y\to 0\\ y\in D_*^2}} \frac{G_{D_*^2}(x,y)}{G_D(x,y)} = 0.$$

Now we deduce as in part (i) of this proof that

$$\lim_{\varepsilon\to 0} \frac{G_{D_*^2}(x,0+\varepsilon N)}{G_D(x,0+\varepsilon N)} = 0.$$

When combined with (4.9) this gives $\partial G_D/\partial N = \infty$. $\square$

REMARK 4.2. The last theorem is a stronger version of Corollaries 4.1, 4.2 and 4.3 of Burdzy [**4**] and generalizes some results of Widman [**20**] and Rippon [**17**].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CALIFORNIA 92093 (Current address of R. J. Williams)

Current address: (Krzysztof Burdzy): Institute of Mathematics, Polish Academy of Sciences, ul. Kopernika 18, 51-617 Wrocław, Poland